Bayesian Nonparametric Inference in Stochastic Volatility Modelling

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Greek Stochastics α':

Monte Carlo: Probability and Methods

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Outline of the talk

- Introduction
- Financial Data
- Stochastic Volatility Model
- Dirichlet Process Mixture model
- Log-Volatility Samplers
- Inference for the parameters
- Results

Introduction

Interest in modelling financial data dates back to 1900, where Bachelier modelled the behavior of a single stock.

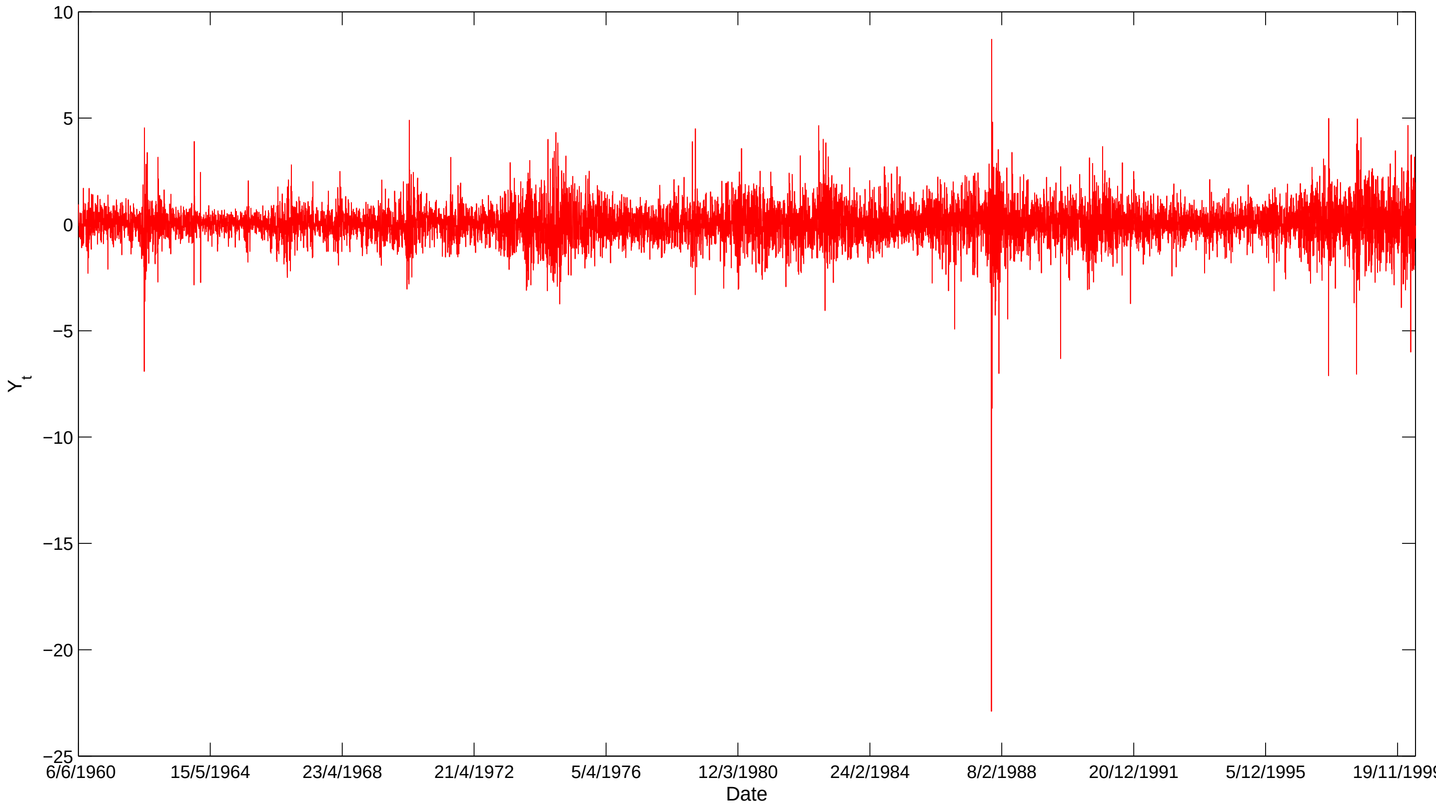
Ever since, many different types of models have been suggested for the modelling of such data.

The models are divided into two groups: the discrete time models and the continuous time models.

Financial Data

Let $\{S_t\}$ be the S&P 500 index and let y_t denote the log returns of this index,

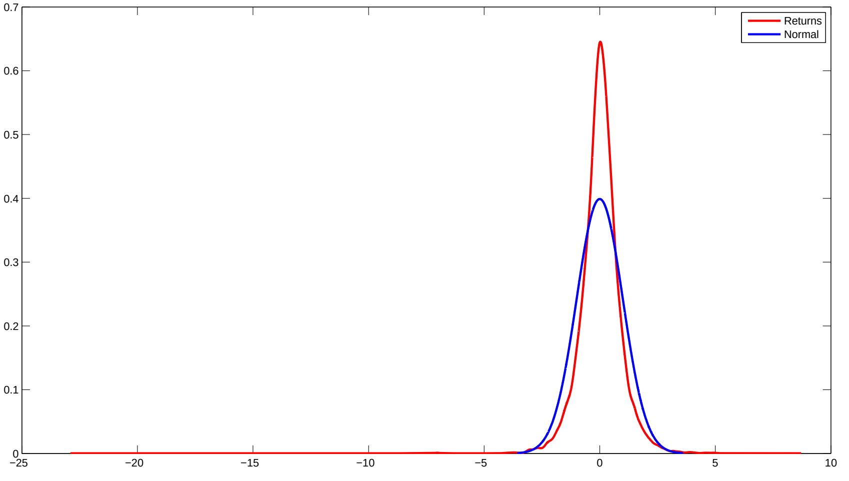
$$y_t = \log(S_t) - \log(S_{t-1}).$$



Stylized Features of Financial Data

The term stylized features refers to the features that describe financial data, which the models must capture (Taylor 1986). Such features are,

- Fat Tails,
- Asymmetry,
- Volatility Clustering,
- Aggregational Gaussianity,
- Seasonality.



Stochastic Volatility Model

Stochastic volatility (SV) models capture the time varying variance of the log-returns of assets.
In the standard discrete SV model, the variance follows a latent stochastic (log-normal autoregressive) process.

The SV model by Taylor (1982) is of the form:

$$y_t = \beta \exp\{h_t/2\}\,\epsilon_t,$$

$$h_t = \mu + \phi\,(h_{t-1} - \mu) + \sigma_\eta \eta_t,$$

where:

- β is the modal instantaneous volatility,
- μ is the log-run mean of the log-volatility,
- $|\phi| < 1$ is the volatility persistence,
- ϵ_t, η_t are uncorrelated standard normal shocks,
- σ_η is the volatility of the log-volatility.

State Space Representation

A representation of the SV model as a dynamic linear model is given by:

$$\log y_t^2 = h_t + \log \epsilon_t^2$$

$$h_t = \mu + \phi\left(h_{t-1} - \mu\right) + \sigma_\eta \eta_t.$$

Equivalently,

$$y_t^* = h_t + z_t$$

$$h_t = \mu + \phi\left(h_{t-1} - \mu\right) + \sigma_\eta \eta_t$$

where $z_t \sim \log \chi_1^2$.

Dirichlet Process Mixture Model

The stochastic volatility model can be reparametrized,

$$y_t^* = h_t^* + z_t^*$$

$$h_t^* = \phi h_{t-1}^* + \sigma_\eta \eta_t$$

where $h_t^* = h_t - \mu$ and $z_t^* = z_t + \mu$.

The error term z_t^* can be modeled as a Dirichlet process mixture model. We use a representation as proposed by Griffin (2009) where the hyperparameters are treated as the location, scale and smoothness of the density,

$$z_t^* | \mu_t \sim N\left(z_t^* | \mu_t, \beta \sigma^2\right)$$

$$\mu_t | G \sim G$$

$$G \sim DP\left(\alpha G_0\right)$$

$$G_0 \equiv N\left(\mu, (1-\beta)\,\sigma^2\right).$$

Offset Mixture Representation

- **Parametric** According to Kim et al. (1998) an offset of seven normal distributions is used to approximate the likelihood. The purpose of this is to have an efficient procedure which samples all the log-volatilities at once. In order to correct the approximation error, a reweighting procedure is used.

- **Nonparametric** The model presented extends this idea by using instead the seven mixture normal, a nonparametric offset mixture. This is done by using nonparametric schemes available in the current literature. Most of the algorithms are based on the Blackwell- MacQueen (1973) representation of the prior distribution as a Pólya urn Scheme.

$$\mu_i|\mu_1, \ldots \mu_{i-1} \sim \frac{1}{i-1+\alpha} \sum_{j=1}^{i-1} \delta\left(\mu_j\right) + \frac{\alpha}{i-1+\alpha} G_0$$

where $\delta\left(\mu_j\right)$ is the distribution concentrated at the single observation μ_j.

Log-Volatility Samplers

- In literature, there have been suggested different ways of estimating the log-volatilities, such as: single-state and multi-state samplers.
- Jacquier et al. (1994) introduced the single-state sampler. Each log-volatility is updated individually using an accept/reject Metropolis-Hastings algorithm.
- Carter and Kohn (1994) and de Jong and Shephard (1995) proposed a multi-state sampler based on the Kalman filter, where the log-volatilies are sampled simultaneously.
- Carter and Kohn (1994) and Frühwirth-Schnatter (1994) introduced an algortihm where the vector of the log-volatilities was updated with the forward filtering backward sampling code (FFBS)

Inference for the parameters of the SV

Parametric Model

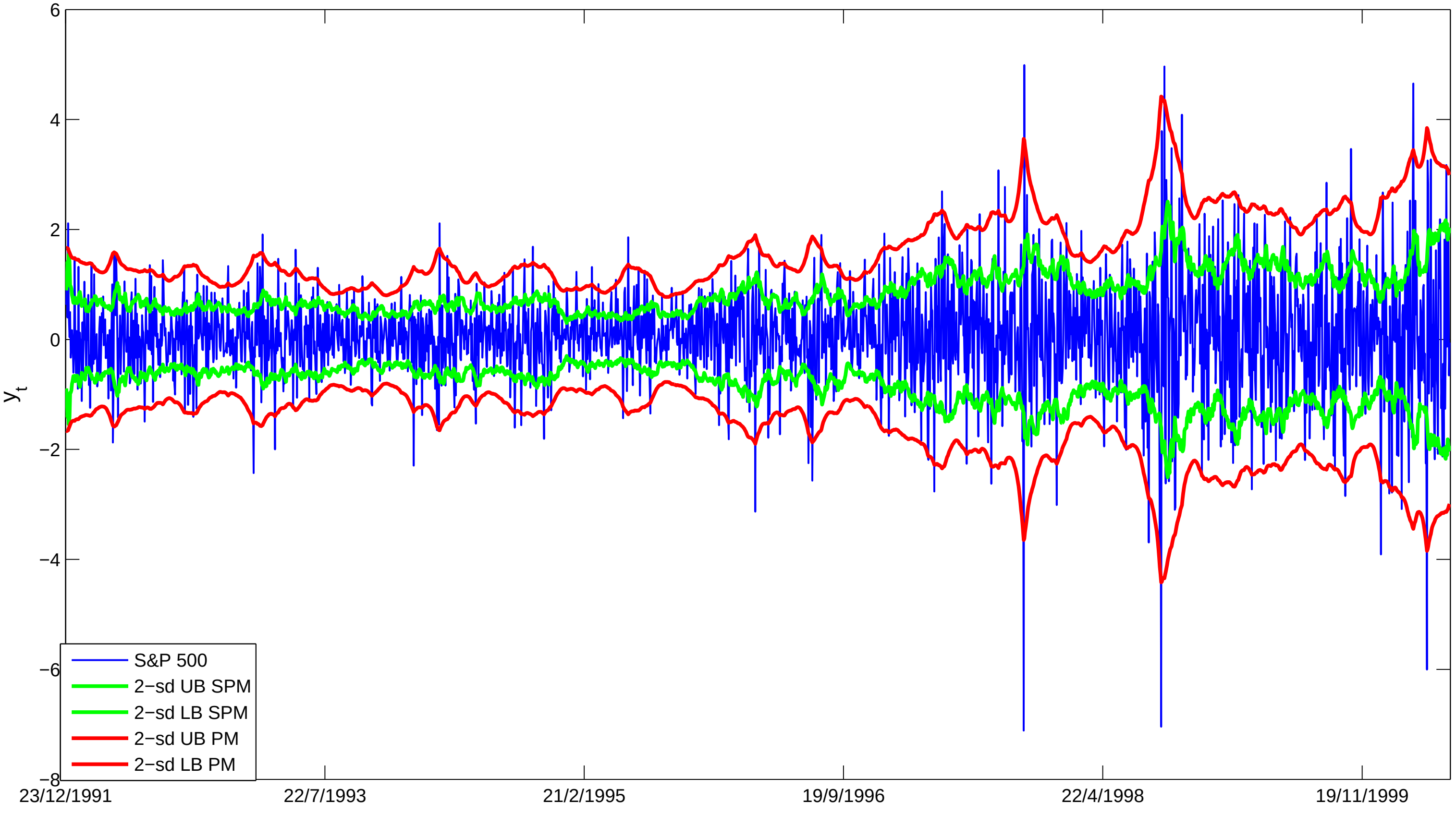
- Initialize s, ϕ, σ_η^2 and μ.
- Draw $\mathbf{h}$ from $\mathbf{h}|\mathbf{y}^*, s, \phi, \sigma_\eta^2, \mu$ using the FFBS.
- Draw s from $s|\mathbf{y}^*, \mathbf{h}$ using the 7-normal mixture model.
- Draw $\phi, \sigma_\eta^2, \mu|\mathbf{h}$ using MCMC schemes.

Semiparametric Model

- Initialize ϕ, σ_η^2, μ and μ_i.
- Draw $\mathbf{h}^*$ from $\mathbf{h}^*|\mathbf{y}^*, s, \phi, \sigma_\eta^2, \mu, \sigma^2$ using the FFBS.
- Draw μ_i using nonparametric schemes.
- Draw ϕ, σ_η^2, μ and σ^2 using MCMC schemes.

Results

The data used are the log-returns of the S&P 500 index for the period of December 12, 1992 to June 6, 2000.



Results

The results for both models are based on the prior distributions:

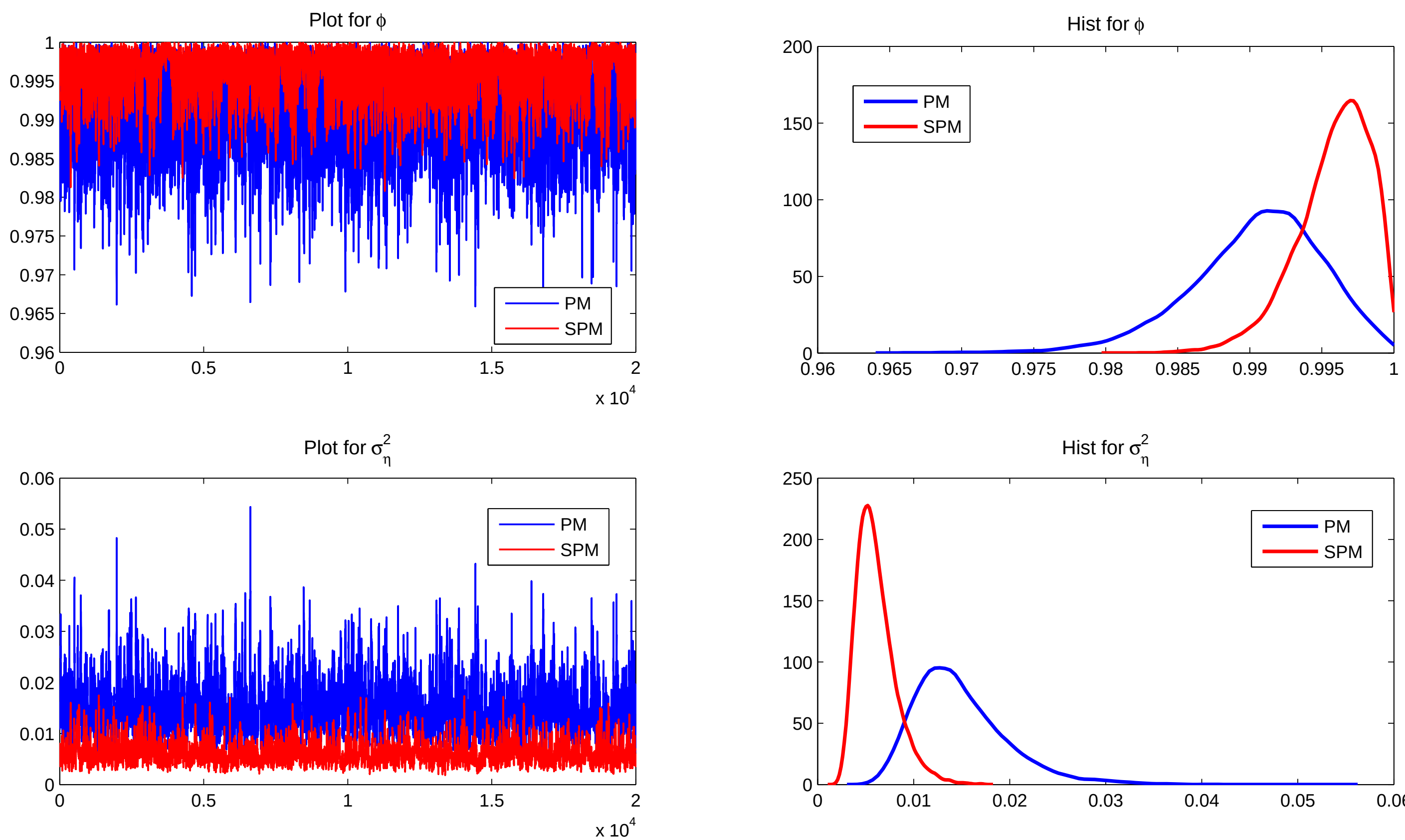
$$\phi \sim N\left(0,10\right) \times I_{\phi}\left(-1,1\right), \sigma_{\eta}^{2} \sim IG\left(2.5,0.0025\right),$$

$$\mu \sim N\left(0,10\right), \sigma^{2} \sim IG\left(2.5,0.0025\right).$$

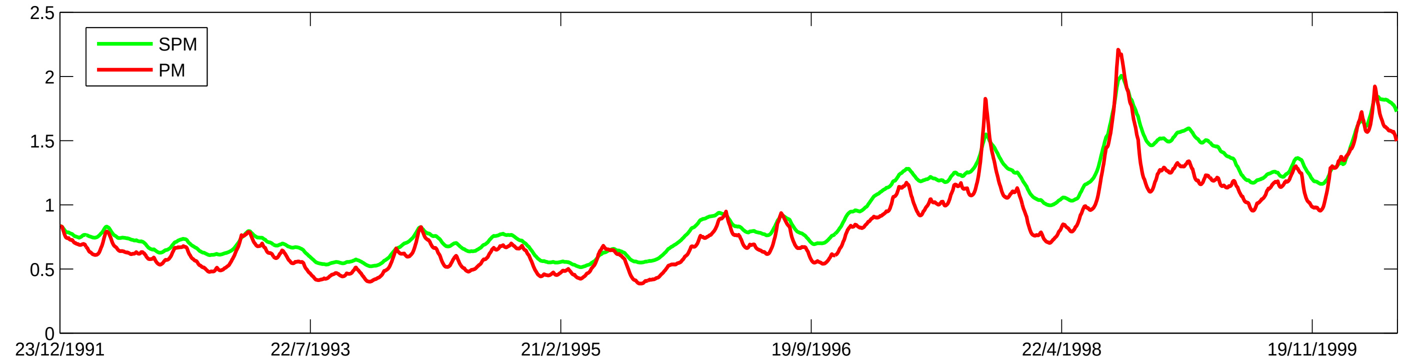
The number of iterations is 210000 of which the first 10000 are discarded as burn-in period. After this period, we apply thinning keeping every 10^{th} draw.

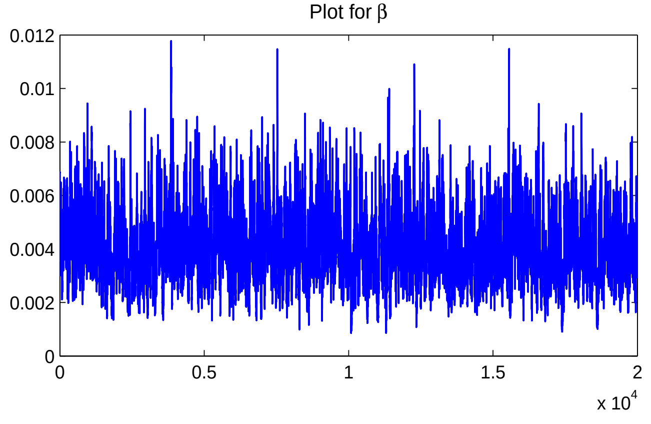
	SPM			**PM**		
	Mean	Std	95%CI	Mean	Std	95%CI
ϕ	0.996	0.003	$[0.991, 0.999]$	0.991	0.004	$[0.982, 0.997]$
σ_{η}^{2}	0.006	0.002	$[0.003, 0.010]$	0.015	0.005	$[0.009, 0.024]$
μ	-1.605	0.5634	$[-2.418, -0.599]$	-0.435	0.640	$[-0.973, 0.164]$
σ^{2}	6.5716	2.5053	$[2.847, 10.483]$			

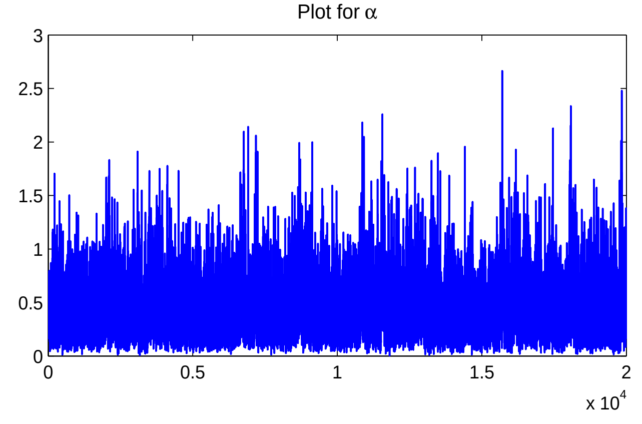
Results


Plot for ϕ
PM
SPM
x 10^4
Hist for ϕ
PM
SPM
Plot for σ_η^2
PM
SPM
x 10^4
Hist for σ_η^2
PM
SPM

Results


SPM
PM
2.5
2
1.5
1
0.5
0
23/12/1991
22/7/1993
21/2/1995
19/9/1996
22/4/1998
19/11/1999


Plot for β
0.012
0.01
0.008
0.006
0.004
0.002
0
0
0.5
1
1.5
2
x 10^4


Plot for α
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
x 10^4

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Thank you!